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Optimal inequalities for bounding Toader mean by arithmetic and quadratic means

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Abstract

In this paper, we present the best possible parameters $\alpha(r)$ and $\beta(r)$ such that the double inequality

$$\begin{aligned} [\alpha(r)A^r(a, b) + (1 - \alpha(r))Q^r(a, b)]^{1/r} &< TD[A(a, b), Q(a, b)] \\ &< [\beta(r)A^r(a, b) + (1 - \beta(r))Q^r(a, b)]^{1/r} \end{aligned}$$

holds for all $r \leq 1$ and $a, b > 0$ with $a \neq b$, and we provide new bounds for the complete elliptic integral $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta$ ($r \in (0, \sqrt{2}/2)$) of the second kind, where $TD(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$, $A(a, b) = (a + b)/2$ and $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ are the Toader, arithmetic, and quadratic means of a and b , respectively.

MSC: 26E60

Keywords: arithmetic mean; Toader mean; quadratic mean; complete elliptic integral

1 Introduction

For $p \in [0, 1]$, $q \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the p th generalized Seiffert mean $S_p(a, b)$, q th Gini mean $G_q(a, b)$, q th power mean $M_q(a, b)$, q th Lehmer mean $L_q(a, b)$, harmonic mean $H(a, b)$, geometric mean $G(a, b)$, arithmetic mean $A(a, b)$, quadratic mean $Q(a, b)$, Toader mean $TD(a, b)$ [1], centroidal mean $\overline{C}(a, b)$, contraharmonic mean $C(a, b)$ are, respectively, defined by

$$\begin{aligned} S_p(a, b) &= \begin{cases} \frac{p(a-b)}{\arctan[2p(a-b)/(a+b)]}, & 0 < p \leq 1, \\ (a+b)/2, & p = 0, \end{cases} \\ G_q(a, b) &= \begin{cases} [(a^{q-1} + b^{q-1})/(a+b)]^{1/(q-2)}, & q \neq 2, \\ (a^a b^b)^{1/(a+b)}, & q = 2, \end{cases} \\ M_q(a, b) &= \begin{cases} [(a^q + b^q)/2]^{1/q}, & q \neq 0, \\ \sqrt{ab}, & q = 0, \end{cases} \end{aligned}$$

$$\begin{aligned}
L_q(a, b) &= \frac{a^{q+1} + b^{q+1}}{a^q + b^q}, & H(a, b) &= \frac{2ab}{a+b}, & G(a, b) &= \sqrt{ab}, \\
A(a, b) &= \frac{a+b}{2}, & Q(a, b) &= \sqrt{\frac{a^2 + b^2}{2}}, \\
TD(a, b) &= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta, \\
\overline{C}(a, b) &= \frac{2(a^2 + ab + b^2)}{3(a+b)}, & C(a, b) &= \frac{a^2 + b^2}{a+b}.
\end{aligned} \tag{1.1}$$

It is well known that $S_p(a, b)$, $G_q(a, b)$, $M_q(a, b)$, and $L_q(a, b)$ are continuous and strictly increasing with respect to $p \in [0, 1]$ and $q \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$\begin{aligned}
H(a, b) &= M_{-1}(a, b) = L_{-1}(a, b) < G(a, b) = M_0(a, b) = L_{-1/2}(a, b) \\
&< A(a, b) = M_1(a, b) = L_0(a, b) < TD(a, b) < \overline{C}(a, b) \\
&< Q(a, b) = M_2(a, b) < C(a, b) = L_1(a, b)
\end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

The Toader mean $TD(a, b)$ has been well known in the mathematical literature for many years, it satisfies

$$TD(a, b) = R_E(a^2, b^2),$$

where

$$R_E(a, b) = \frac{1}{\pi} \int_0^\infty \frac{[a(t+b) + b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}} dt$$

stands for the symmetric complete elliptic integral of the second kind (see [2–4]), therefore it cannot be expressed in terms of the elementary transcendental functions.

Let $r \in (0, 1)$, $\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta$ and $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta$ be, respectively, the complete elliptic integrals of the first and second kind. Then $\mathcal{K}(0^+) = \mathcal{E}(0^+) = \pi/2$, $\mathcal{K}(r)$, and $\mathcal{E}(r)$ satisfy the derivatives formulas (see [5], Appendix E, p.474–475)

$$\begin{aligned}
\frac{d\mathcal{K}(r)}{dr} &= \frac{\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)}{r(1 - r^2)}, & \frac{d\mathcal{E}(r)}{dr} &= \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \\
\frac{d[\mathcal{K}(r) - \mathcal{E}(r)]}{dr} &= \frac{r\mathcal{E}(r)}{1 - r^2},
\end{aligned}$$

the values $\mathcal{K}(\sqrt{2}/2)$ and $\mathcal{E}(\sqrt{2}/2)$ can be expressed as (see [6], Theorem 1.7)

$$\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}} = 1.854\dots, \quad \mathcal{E}\left(\frac{\sqrt{2}}{2}\right) = \frac{4\Gamma^2(3/4) + \Gamma^2(1/4)}{8\sqrt{\pi}} = 1.350\dots,$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ ($\operatorname{Re} x > 0$) is the Euler gamma function, and the Toader mean $TD(a, b)$ can be rewritten as

$$TD(a, b) = \begin{cases} 2a\mathcal{E}(\sqrt{1 - (b/a)^2})/\pi, & a \geq b, \\ 2b\mathcal{E}(\sqrt{1 - (a/b)^2})/\pi, & a < b. \end{cases} \quad (1.2)$$

Recently, the Toader mean $TD(a, b)$ has been the subject of intensive research. Vuorinen [7] conjectured that the inequality

$$TD(a, b) > M_{3/2}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [8], and Barnard, Pearce and Richards [9], respectively.

Alzer and Qiu [10] presented a best possible upper power mean bound for the Toader mean as follows:

$$TD(a, b) < M_{\log 2 / (\log \pi - \log 2)}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

Neuman [2], and Kazi and Neuman [3] proved that the inequalities

$$\frac{(a+b)\sqrt{ab} - ab}{AGM(a, b)} < TD(a, b) < \frac{4(a+b)\sqrt{ab} + (a-b)^2}{8AGM(a, b)},$$

$$TD(a, b) < \frac{1}{4} \left(\sqrt{(2 + \sqrt{2})a^2 + (2 - \sqrt{2})b^2} + \sqrt{(2 + \sqrt{2})b^2 + (2 - \sqrt{2})a^2} \right)$$

hold for all $a, b > 0$ with $a \neq b$, where $AGM(a, b)$ is the arithmetic-geometric mean of a and b .

In [11–13], the authors presented the best possible parameters $\lambda_1, \mu_1 \in [0, 1]$ and $\lambda_2, \mu_2, \lambda_3, \mu_3 \in \mathbb{R}$ such that the double inequalities $S_{\lambda_1}(a, b) < TD(a, b) < S_{\mu_1}(a, b)$, $G_{\lambda_2}(a, b) < TD(a, b) < G_{\mu_2}(a, b)$ and $L_{\lambda_3}(a, b) < TD(a, b) < L_{\mu_3}(a, b)$ hold for all $a, b > 0$ with $a \neq b$.

Let $\lambda, \mu, \alpha, \beta \in (1/2, 1)$. Then Chu, Wang and Ma [14], and Hua and Qi [15] proved that the double inequalities

$$C[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a] < TD(a, b) < C[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a],$$

$$\overline{C}[\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a] < TD(a, b) < \overline{C}[\beta a + (1 - \beta)b, \beta b + (1 - \beta)a]$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq 3/4$, $\mu \geq 1/2 + \sqrt{\pi(4 - \pi)}/(2\pi)$, $\alpha \leq 1/2 + \sqrt{3}/4$ and $\beta \geq 1/2 + \sqrt{12/\pi - 3}/2$.

In [16–20], the authors proved that the double inequalities

$$\alpha_1 Q(a, b) + (1 - \alpha_1)A(a, b) < TD(a, b) < \beta_1 Q(a, b) + (1 - \beta_1)A(a, b),$$

$$Q^{\alpha_2}(a, b)A^{(1 - \alpha_2)}(a, b) < TD(a, b) < Q^{\beta_2}(a, b)A^{(1 - \beta_2)}(a, b),$$

$$\alpha_3 C(a, b) + (1 - \alpha_3)A(a, b) < TD(a, b) < \beta_3 C(a, b) + (1 - \beta_3)A(a, b),$$

$$\begin{aligned}
& \frac{\alpha_4}{A(a,b)} + \frac{1-\alpha_4}{C(a,b)} < \frac{1}{TD(a,b)} < \frac{\beta_4}{A(a,b)} + \frac{1-\beta_4}{C(a,b)}, \\
& \alpha_5 C(a,b) + (1-\alpha_5)H(a,b) < TD(a,b) < \beta_5 C(a,b) + (1-\beta_5)H(a,b), \\
& \alpha_6 [C(a,b) - H(a,b)] + A(a,b) < TD(a,b) < \beta_6 [C(a,b) - H(a,b)] + A(a,b), \\
& \alpha_7 \overline{C}(a,b) + (1-\alpha_7)A(a,b) < TD(a,b) < \beta_7 \overline{C}(a,b) + (1-\beta_7)A(a,b), \\
& \frac{\alpha_8}{A(a,b)} + \frac{1-\alpha_8}{\overline{C}(a,b)} < \frac{1}{TD(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1-\beta_8}{\overline{C}(a,b)}, \\
& \alpha_9 Q(a,b) + (1-\alpha_9)H(a,b) < TD(a,b) < \beta_9 Q(a,b) + (1-\beta_9)H(a,b), \\
& \frac{\alpha_{10}}{H(a,b)} + \frac{1-\alpha_{10}}{Q(a,b)} < \frac{1}{TD(a,b)} < \frac{\beta_{10}}{H(a,b)} + \frac{1-\beta_{10}}{Q(a,b)}
\end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/2$, $\beta_1 \geq (4 - \pi)/[(\sqrt{2} - 1)\pi]$, $\alpha_2 \leq 1/2$, $\beta_2 \geq 4 - 2 \log \pi / \log 2$, $\alpha_3 \leq 1/4$, $\beta_3 \geq 4/\pi - 1$, $\alpha_4 \leq \pi/2 - 1$, $\beta_4 \geq 3/4$, $\alpha_5 \leq 5/8$, $\beta_5 \geq 2/\pi$, $\alpha_6 \leq 1/8$, $\beta_6 \geq 2/\pi - 1/2$, $\alpha_7 \leq 3/4$, $\beta_7 \geq 12/\pi - 3$, $\alpha_8 \leq \pi - 3$, $\beta_8 \geq 1/4$, $\alpha_9 \leq 5/6$, $\beta_9 \geq 2\sqrt{2}/\pi$, $\alpha_{10} \leq 0$, and $\beta_{10} \geq 1/6$.

The main purpose of this paper is to present the best possible parameters $\alpha(r)$ and $\beta(r)$ such that the double inequality

$$\begin{aligned}
& [\alpha(r)A^r(a,b) + (1-\alpha(r))Q^r(a,b)]^{1/r} < TD[A(a,b), Q(a,b)] \\
& < [\beta(r)A^r(a,b) + (1-\beta(r))Q^r(a,b)]^{1/r}
\end{aligned}$$

holds for all $r \leq 1$ and $a, b > 0$ with $a \neq b$.

2 Lemmas

In order to prove our main result we need two lemmas, which we present in this section.

Lemma 2.1 Let $p \in (0, 1)$, $t \in (0, \sqrt{2}/2)$, $\lambda = (2 + \sqrt{2})[1 - 2\mathcal{E}(\sqrt{2}/2)/\pi] = 0.478\dots$ and

$$f(t) = \frac{\pi p}{2} \sqrt{1-t^2} + \frac{\pi}{2} (1-p) - \mathcal{E}(t). \quad (2.1)$$

Then $f(t) < 0$ for all $t \in (0, \sqrt{2}/2)$ if and only if $p \geq 1/2$ and $f(t) > 0$ for all $t \in (0, \sqrt{2}/2)$ if and only if $p \leq \lambda$.

Proof It follows from (2.1) that

$$f(0^+) = 0, \quad (2.2)$$

$$f\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2}\right) (\lambda - p), \quad (2.3)$$

$$f'(t) = \frac{f_1(t)}{t\sqrt{1-t^2}}, \quad (2.4)$$

where

$$f_1(t) = \sqrt{1-t^2} [\mathcal{K}(t) - \mathcal{E}(t)] - \frac{\pi p}{2} t^2, \quad f_1(0^+) = 0, \quad (2.5)$$

$$f_1\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{E}\left(\frac{\sqrt{2}}{2}\right) \right] - \frac{\pi p}{4}, \quad (2.6)$$

$$f_1'(t) = \frac{t[2\mathcal{E}(t) - \mathcal{K}(t)]}{\sqrt{1-t^2}} - \pi p t, \quad (2.7)$$

$$f_1'(0^+) = 0, \quad (2.8)$$

$$f_1'\left(\frac{\sqrt{2}}{2}\right) = 2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}\pi p}{2}, \quad (2.9)$$

$$f_1''(t) = \frac{(3-2t^2)\mathcal{E}(t) - (2-t^2)\mathcal{K}(t)}{(1-t^2)^{3/2}} - \pi p, \quad (2.10)$$

$$f_1''(0^+) = \pi\left(\frac{1}{2} - p\right), \quad (2.11)$$

$$f_1''\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \left[4\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - 3\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) \right] - \pi p, \quad (2.12)$$

$$f_1'''(t) = -\frac{(1+t^2)[\mathcal{K}(t) - \mathcal{E}(t)] + t^2\mathcal{K}(t)}{t(1-t^2)^{5/2}} < 0 \quad (2.13)$$

for all $t \in (0, \sqrt{2}/2)$.

It follows from (2.13) that $f_1''(t)$ is strictly decreasing on $(0, \sqrt{2}/2)$.

We divide the proof into three cases.

Case 1 $p \geq 1/2$. Then (2.11) leads to

$$f_1''(0^+) \leq 0. \quad (2.14)$$

From (2.14) and the monotonicity of $f_1''(t)$ we clearly see that $f_1'(t)$ is strictly decreasing on $(0, \sqrt{2}/2)$. Therefore, $f(t) < 0$ for all $t \in (0, \sqrt{2}/2)$ follows easily from (2.2), (2.4), (2.5), (2.8), and the monotonicity of $f_1'(t)$.

Case 2 $0 < p \leq \lambda$. Then from (2.11) and (2.12) together with $4\mathcal{E}(\sqrt{2}/2) - 3\mathcal{K}(\sqrt{2}/2) = -0.159\dots$ we clearly see that

$$f_1''(0^+) > 0, \quad f_1''\left(\frac{\sqrt{2}}{2}\right) < 0. \quad (2.15)$$

It follows from (2.15) and the monotonicity of $f_1''(t)$ that there exists $t_0 \in (0, \sqrt{2}/2)$ such that $f_1'(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, \sqrt{2}/2)$.

Let $\lambda^* = \frac{\sqrt{2}}{\pi} [2\mathcal{E}(\frac{\sqrt{2}}{2}) - \mathcal{K}(\frac{\sqrt{2}}{2})] = 0.381\dots$ and $\lambda^{**} = \frac{2\sqrt{2}}{\pi} [\mathcal{K}(\frac{\sqrt{2}}{2}) - \mathcal{E}(\frac{\sqrt{2}}{2})] = 0.453\dots$. We divide the proof into three subcases.

Subcase 2.1 $0 < p \leq \lambda^*$. Then (2.9) leads to

$$f_1'\left(\frac{\sqrt{2}}{2}\right) \geq 0. \quad (2.16)$$

It follows from (2.8) and (2.16) together with the piecewise monotonicity of $f_1'(t)$ that

$$f_1'(t) > 0 \quad (2.17)$$

for all $t \in (0, \sqrt{2}/2)$.

Therefore, $f(t) > 0$ for all $t \in (0, \sqrt{2}/2)$ follows easily from (2.2), (2.4), (2.5), and (2.17).

Subcase 2.2 $\lambda^* < p \leq \lambda^{**}$. Then (2.6) and (2.9) lead to

$$f_1\left(\frac{\sqrt{2}}{2}\right) \geq 0, \quad (2.18)$$

$$f_1'\left(\frac{\sqrt{2}}{2}\right) < 0. \quad (2.19)$$

It follows from (2.8) and (2.19) together with the piecewise monotonicity of $f_1'(t)$ that there exists $t_1 \in (0, \sqrt{2}/2)$ such that $f_1(t)$ is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, \sqrt{2}/2)$.

Equation (2.5) and inequality (2.18) together with the piecewise monotonicity of $f_1(t)$ lead to the conclusion that

$$f_1(t) > 0 \quad (2.20)$$

for all $t \in (0, \sqrt{2}/2)$.

Therefore, $f(t) > 0$ for all $t \in (0, \sqrt{2}/2)$ follows easily from (2.2) and (2.4) together with (2.20).

Subcase 2.3 $\lambda^{**} < p \leq \lambda$. Then (2.3), (2.6), and (2.9) lead to

$$f\left(\frac{\sqrt{2}}{2}\right) \geq 0, \quad (2.21)$$

$$f_1\left(\frac{\sqrt{2}}{2}\right) < 0, \quad (2.22)$$

$$\begin{aligned} f_1'\left(\frac{\sqrt{2}}{2}\right) &< 2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}\pi}{2}\lambda^{**} \\ &< 2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}\pi}{2}\lambda^* = 0. \end{aligned} \quad (2.23)$$

It follows from (2.8) and (2.23) together with the piecewise monotonicity of $f_1'(t)$ that there exists $t_2 \in (0, \sqrt{2}/2)$ such that $f_1(t)$ is strictly increasing on $(0, t_2]$ and strictly decreasing on $[t_2, \sqrt{2}/2)$.

From (2.4), (2.5), and (2.22) together with the piecewise monotonicity of $f_1(t)$ we clearly see that there exists $t_3 \in (0, \sqrt{2}/2)$ such that $f(t)$ is strictly increasing on $(0, t_3]$ and strictly decreasing on $[t_3, \sqrt{2}/2)$.

Therefore, $f(t) > 0$ for all $t \in (0, \sqrt{2}/2)$ follows easily from (2.2) and (2.21) together with the piecewise monotonicity of $f(t)$.

Case 3 $\lambda < p < 1/2$. Then (2.3), (2.6), (2.9), (2.11), and (2.12) lead to

$$f\left(\frac{\sqrt{2}}{2}\right) < 0, \quad (2.24)$$

$$f_1\left(\frac{\sqrt{2}}{2}\right) < \frac{\sqrt{2}}{2} \left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{E}\left(\frac{\sqrt{2}}{2}\right) \right] - \frac{\pi\lambda^{**}}{4} = 0, \quad (2.25)$$

$$f_1'\left(\frac{\sqrt{2}}{2}\right) < 2\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}\pi\lambda^*}{2} = 0, \quad (2.26)$$

$$f_1''(0^+) > 0, \quad (2.27)$$

$$\begin{aligned} f_1''\left(\frac{\sqrt{2}}{2}\right) &< \sqrt{2}\left[4\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) - 3\mathcal{K}\left(\frac{\sqrt{2}}{2}\right)\right] - \pi\lambda^* \\ &= -2\sqrt{2}\left[\mathcal{K}\left(\frac{\sqrt{2}}{2}\right) - \mathcal{E}\left(\frac{\sqrt{2}}{2}\right)\right] < 0. \end{aligned} \quad (2.28)$$

It follows from (2.27) and (2.28) together with the monotonicity of $f_1'''(t)$ that there exists $t_4 \in (0, \sqrt{2}/2)$ such that $f_1'(t)$ is strictly increasing on $(0, t_4]$ and strictly decreasing on $[t_4, \sqrt{2}/2)$.

Equation (2.8) and inequality (2.26) together with the piecewise monotonicity of $f_1'(t)$ lead to the conclusion that there exists $t_5 \in (0, \sqrt{2}/2)$ such that $f_1(t)$ is strictly increasing on $(0, t_5]$ and strictly decreasing on $[t_5, \sqrt{2}/2)$.

From (2.4), (2.5), (2.25), and the piecewise monotonicity of $f_1(t)$ we clearly see that there exists $t_6 \in (0, \sqrt{2}/2)$ such that $f(t)$ is strictly increasing on $(0, t_6]$ and strictly decreasing on $[t_6, \sqrt{2}/2)$.

Therefore, there exists $t_7 \in (0, \sqrt{2}/2)$ such that $f(t) > 0$ for $t \in (0, t_7)$ and $f(t) < 0$ for $t \in (t_7, \sqrt{2}/2)$ follows from (2.2) and (2.24) together with the piecewise monotonicity of $f(t)$. \square

Lemma 2.2 Let $r \in \mathbb{R}$, $a, b > 0$ with $1 < b/a < \sqrt{2}$, $c_0 = 2\mathcal{E}(\sqrt{2}/2)/\pi = 0.859\dots$, $c_1 = \sqrt{2}/2$, $\lambda(r)$ and $U(r; a, b)$ be defined by

$$\lambda(r) = \frac{1 - c_0^r}{1 - c_1^r} \quad (r \neq 0), \quad \lambda_0 = \frac{\log c_0}{\log c_1}, \quad (2.29)$$

and

$$U(r; a, b) = [\lambda(r)a^r + (1 - \lambda(r))b^r]^{1/r} \quad (r \neq 0), \quad U(0; a, b) = a^{\lambda_0} b^{1-\lambda_0}, \quad (2.30)$$

respectively. Then the function $r \mapsto U(r; a, b)$ is strictly decreasing on $(-\infty, \infty)$.

Proof Let $x = b/a \in (1, \sqrt{2})$, $r \neq 0$, and

$$V(r, x) = (1 - \lambda(r)) \log x - (\log \lambda(r))'. \quad (2.31)$$

Then from (2.29)-(2.31) one has

$$\begin{aligned} \log U(r; a, b) &= \log a + \frac{1}{r} \log(\lambda(r) + (1 - \lambda(r))x^r), \\ \frac{\partial \log U(r; a, b)}{\partial r} &= \frac{\lambda'(r)(1 - x^r) + (1 - \lambda(r))x^r \log x}{r(\lambda(r) + (1 - \lambda(r))x^r)} - \frac{\log(\lambda(r) + (1 - \lambda(r))x^r)}{r^2}, \\ \frac{\partial \log U(r; a, b)}{\partial r} \Big|_{x=1} &= 0, \\ \lambda'(r) &= \frac{(c_1^r - 1)c_0^r \log c_0 - (c_0^r - 1)c_1^r \log c_1}{(c_1^r - 1)^2}, \\ (\lambda(r) + (1 - \lambda(r))x^r)|_{x=\sqrt{2}} &= \frac{1 - c_0^r}{1 - c_1^r} + \left(1 - \frac{1 - c_0^r}{1 - c_1^r}\right) \frac{1}{c_1^r} = \frac{c_0^r}{c_1^r}, \end{aligned} \quad (2.32)$$

$$\left. \frac{\lambda'(r)(1-x^r) + (1-\lambda(r))x^r \log x}{r(\lambda(r) + (1-\lambda(r))x^r)} \right|_{x=\sqrt{2}} = \frac{1}{r} \log \frac{c_0}{c_1},$$

$$\left. \frac{\partial \log U(r; a, b)}{\partial r} \right|_{x=\sqrt{2}} = 0, \quad (2.33)$$

$$\frac{\partial^2 \log U(r; a, b)}{\partial x \partial r} = \frac{\lambda(r)x^{r-1}}{(\lambda(r) + (1-\lambda(r))x^r)^2} V(r, x), \quad (2.34)$$

$$V(r, 1) = \frac{\log \frac{1}{c_1}}{(\frac{1}{c_1})^r - 1} - \frac{\log \frac{1}{c_0}}{(\frac{1}{c_0})^r - 1} < 0, \quad (2.35)$$

$$V(r, \sqrt{2}) = c_0^r \left(\frac{\log c_1}{c_1^r - 1} - \frac{\log c_0}{c_0^r - 1} \right) > 0, \quad (2.36)$$

where inequalities (2.35) and (2.36) hold due to $c_0 > c_1$ and the function $t \mapsto \log t/(t^r - 1)$ is strictly decreasing on $(0, \infty)$.

Note that $\lambda(r) \in (0, 1)$ and the function $x \rightarrow V(r, x)$ is strictly increasing on $(1, \sqrt{2})$. Then (2.34)-(2.36) lead to the conclusion that there exists $x_0 \in (1, \sqrt{2})$ such that the function $x \mapsto \partial \log U(r; a, b)/\partial r$ is strictly decreasing on $(1, x_0)$ and strictly increasing on $(x_0, \sqrt{2})$.

It follows from (2.32) and (2.33) together with the piecewise monotonicity of the function $x \mapsto \partial \log U(r; a, b)/\partial r$ on the interval $(1, \sqrt{2})$ that

$$\frac{\partial \log U(r; a, b)}{\partial r} < 0 \quad (2.37)$$

for all $a, b > 0$ with $1 < b/a < \sqrt{2}$.

Therefore, Lemma 2.2 follows from (2.37). \square

3 Main result

Theorem 3.1 Let $c_0 = 2\mathcal{E}(\sqrt{2}/2)/\pi = 0.859\dots$, $c_1 = \sqrt{2}/2$ and $\lambda(r)$ be defined by (2.29). Then the double inequality

$$\begin{aligned} [\alpha(r)A^r(a, b) + (1 - \alpha(r))Q^r(a, b)]^{1/r} &< TD[A(a, b), Q(a, b)] \\ &< [\beta(r)A^r(a, b) + (1 - \beta(r))Q^r(a, b)]^{1/r} \end{aligned}$$

holds for all $r \leq 1$ and $a, b > 0$ with $a \neq b$ if and only if $\alpha(r) \geq 1/2$ and $\beta(r) \leq \lambda(r)$, where $r = 0$ is the limit value of $r \rightarrow 0$.

Proof We first prove that Theorem 3.1 holds for $r = 1$.

Since $A(a, b) < TD[A(a, b), Q(a, b)] < Q(a, b)$ for all $a, b > 0$ with $a \neq b$, and $A(a, b)$, $TD(a, b)$ and $Q(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $\alpha(1)$, $\beta(1) \in (0, 1)$ and $a > b$. Let $t = (a - b)/\sqrt{2(a^2 + b^2)} \in (0, \sqrt{2}/2)$ and $p \in (0, 1)$. Then (1.1) and (1.2) lead to

$$\begin{aligned} A(a, b) &= Q(a, b)\sqrt{1 - t^2}, \quad TD[A(a, b), Q(a, b)] = \frac{2}{\pi} Q(a, b)\mathcal{E}(t), \\ pA(a, b) + (1 - p)Q(a, b) - TD[A(a, b), Q(a, b)] &= \frac{2}{\pi} Q(a, b)f(t), \end{aligned} \quad (3.1)$$

where $f(t)$ is defined as in Lemma 2.1.

Therefore, Theorem 3.1 for $r = 1$ follows easily from Lemma 2.1 and (3.1).

Next, let $r < 1$ and $a, b > 0$ with $a \neq b$, then it follows from Theorem 3.1 for $r = 1$ that

$$\frac{A(a, b) + Q(a, b)}{2} < TD[A(a, b), Q(a, b)] < \lambda(1)A(a, b) + (1 - \lambda(1))Q(a, b). \quad (3.2)$$

Note that

$$1 < \frac{Q(a, b)}{A(a, b)} < \sqrt{2}, \quad (3.3)$$

$$\frac{TD[A(a, b), Q(a, b)]}{\left[\frac{A^r(a, b) + Q^r(a, b)}{2}\right]^{1/r}} = \frac{2^{1+1/r}}{\pi} \frac{\mathcal{E}(t)}{[1 + (1 - t^2)^{r/2}]^{1/r}}, \quad (3.4)$$

$$\frac{TD[A(a, b), Q(a, b)]}{[\lambda(r)A^r(a, b) + (1 - \lambda(r))Q^r(a, b)]^{1/r}} = \frac{2}{\pi} \frac{\mathcal{E}(t)}{[\lambda(r)(1 - t^2)^{r/2} + 1 - \lambda(r)]^{1/r}}, \quad (3.5)$$

$$\lim_{t \rightarrow 0^+} \frac{2^{1+1/r}}{\pi} \frac{\mathcal{E}(t)}{[1 + (1 - t^2)^{r/2}]^{1/r}} = \lim_{t \rightarrow \sqrt{2}/2} \frac{2}{\pi} \frac{\mathcal{E}(t)}{[\lambda(r)(1 - t^2)^{r/2} + 1 - \lambda(r)]^{1/r}} = 1. \quad (3.6)$$

Therefore, Theorem 3.1 for $r < 1$ follows from (3.2)-(3.6) and Lemma 2.2 together with the monotonicity of the function $r \mapsto [(a^r + b^r)/2]^{1/r}$. \square

Let $r = 1$. Then Theorems 3.1 leads to Corollary 3.2 immediately.

Corollary 3.2 *Let $\lambda = (2 + \sqrt{2})[1 - 2\mathcal{E}(\sqrt{2}/2)/\pi]$. Then the double inequality*

$$\frac{\pi}{4} \sqrt{1 - t^2} + \frac{\pi}{4} < \mathcal{E}(t) < \frac{\pi}{2} \lambda \sqrt{1 - t^2} + \frac{\pi}{2} (1 - \lambda)$$

holds for all $t \in (0, \sqrt{2}/2)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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